Jeffrey Seguritan

Calculus – Differentiation

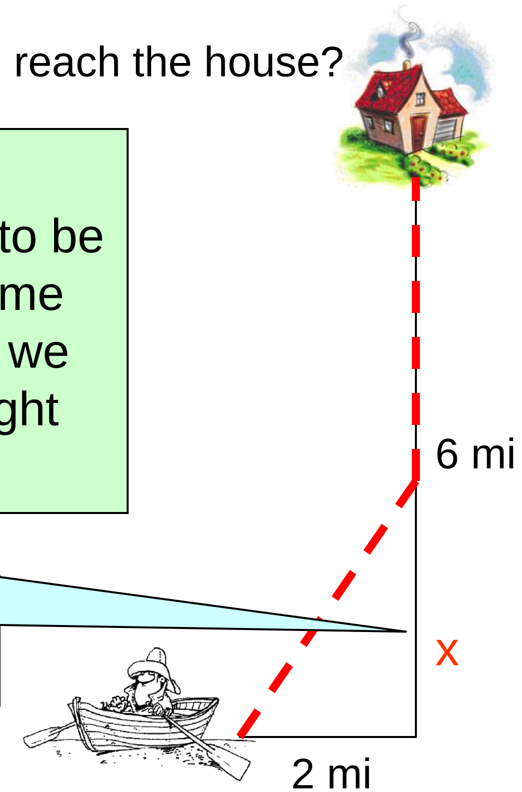
A person in a rowboat 2 miles from the nearest point on a strait shoreline wishes to reach a house 6 miles farther down the shore. The person can row at a rate of 3 mi/hr and walk at a rate of 5 mi/hr.

A) What route should the person take to minimize the amount of time it takes to reach the house?

B) What is the minimum amount of time (in minutes) required to reach the house?

With these type of optimization problems, you always want an equation to optimize. Sometimes, you have to be creative in how you design this equation based on some variable you select as relevant to the problem. Here, we know that the shortest distance to the shore is a straight line to the house BUT it's faster to walk than to row.

Let's make our variable this distance since it relates the both the distance rowed and the distance walked



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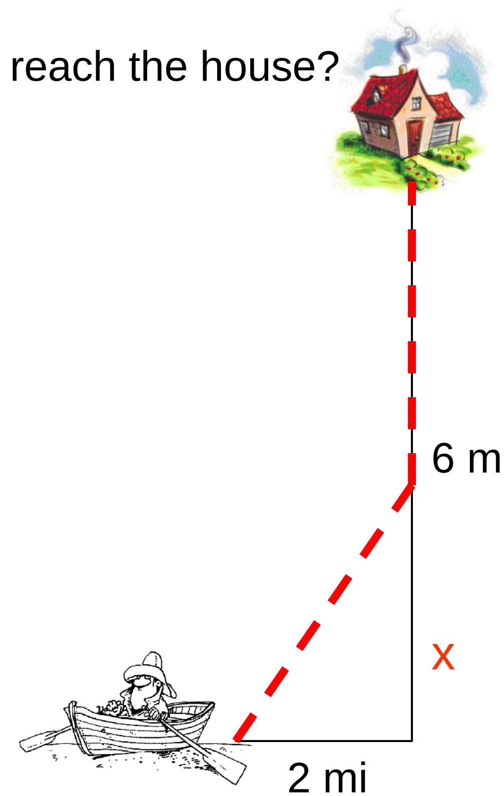

Let's write a function D(x) of the distance traveled depending on the x variable we had selected:

$$D(x) = \sqrt{2^2 + x^2} + (6 - x)$$

But what we really need is to write a function for time, using Time = Distance / Rate:

$$T(x) = \frac{\sqrt{2^2 + x^2}}{3} + \frac{(6 - x)}{5}$$

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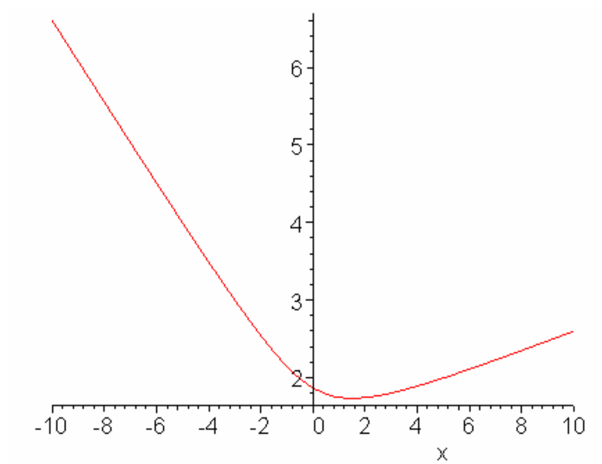
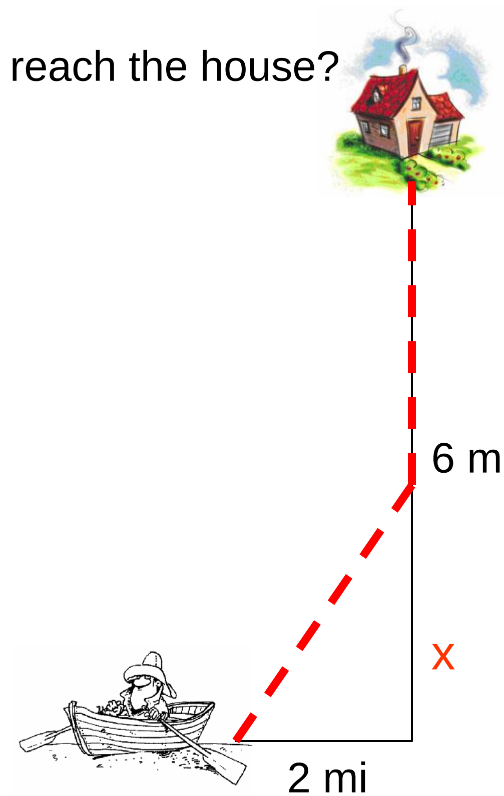
Let's take a look at this Time function to make sure there's a minimum to look for:

$$T(x) = \frac{\sqrt{2^2 + x^2}}{3} + \frac{(6-x)}{5}$$





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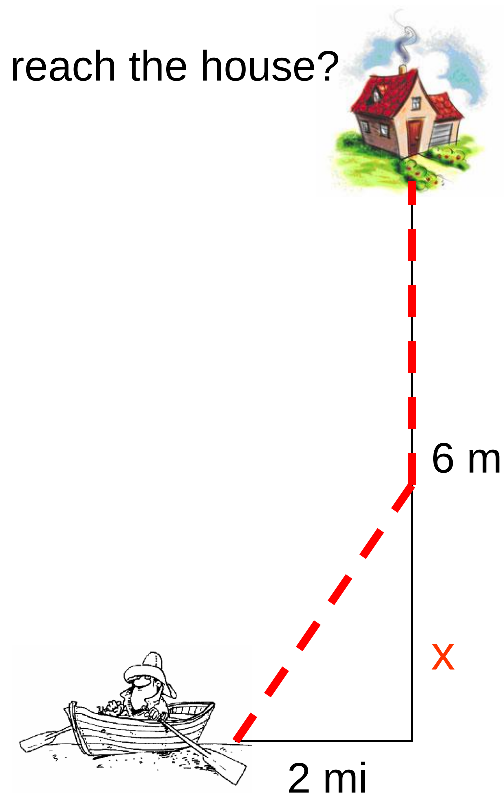
Now let's take the derivative of the Time function:

$$\frac{dT}{dx} = \frac{1}{3}\frac{1}{2}\frac{2x}{\sqrt{2^2+x^2}} - \frac{1}{5}$$

$$\frac{dT}{dx} = \frac{x}{3\sqrt{2^2+x^2}} - \frac{1}{5}$$

$$\frac{dT}{dx} = \frac{5x-3\sqrt{2^2+x^2}}{15\sqrt{2^2+x^2}}$$



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Now let's find the zero of the Time function:

$$\frac{dT}{dx} = \frac{5x - 3\sqrt{2^2 + x^2}}{15\sqrt{2^2 + x^2}} = 0$$

$$5x - 3\sqrt{2^2 + x^2} = 0$$

$$5x = 3\sqrt{2^2 + x^2}$$

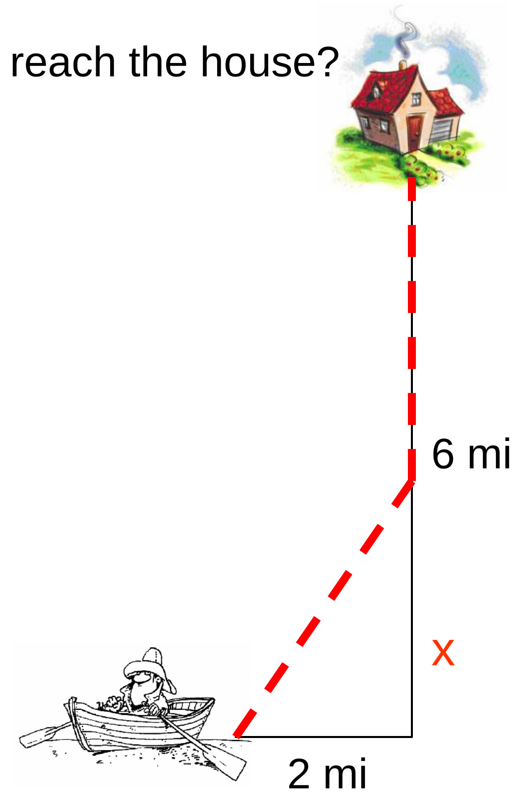
$$25x^2 = 9(4 + x^2)$$

$$25x^2 = 36 + 9x^2$$

$$16x^2 = 36$$

$$x = 3/2$$

This x is a critical value and should represent a minimum in our Time function.



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A) What route should the person take to minimize the amount of time it takes to reach the house?

B) What is the minimum amount of time (in minutes) required to reach the house?

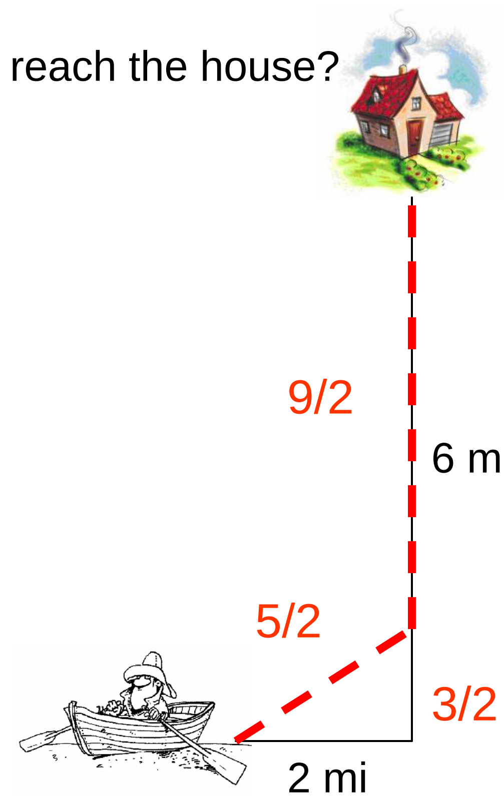
If x = 3/2 mi, then we can find the optimal (shortest time) route:

- 5/2 mi rowing on boat, 9/2 mi walking on the shore

Then the minimum amount of time (in minutes) required to reach the house is:

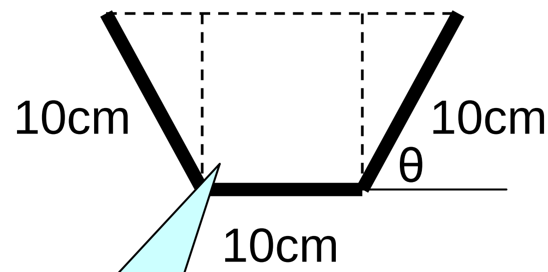
- rowing time = (5/2 mi) / (3 mi/hr) = 5/6 hr = 50 min
- walking time = (9/2 mi) / (5 mi/hr) = 9/10 hr = 54 min

Total time = 50 min + 54 min = 104 min



Calculus – Differentiation

A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle. What should the measure of the angle be so that the gutter will carry the maximum amount of water?



So for this problem, let's find a function for the enclosed area of the gutter which depends on the angle:

Notice that this area can be broken up into 1 rectangle and 2 right triangles

Area of Rectangle

Area of 2 Right triangles

$$A(q) = (10)(10\sin q) + 2\cdot\frac{1}{2}(10\cos q)(10\sin q)$$

$$A(q) = (10)(10\sin q) + (10\cos q)(10\sin q)$$

$$A(q) = 100\sin q + 100\cos q\,\sin q$$

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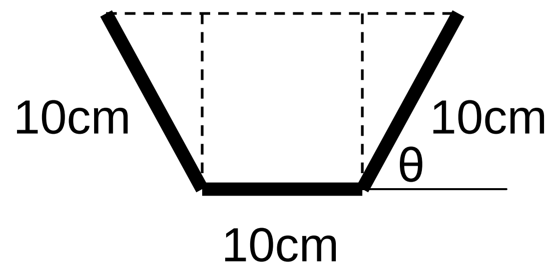
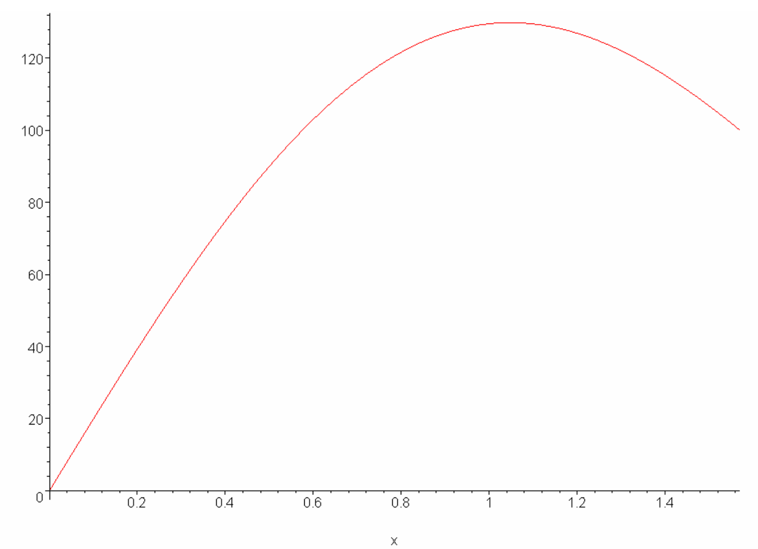


Let's make sure there's a maximum to the area function we wrote, and we see that there is one:

$$A(q) = 100\sin q + 100\cos q \sin q$$



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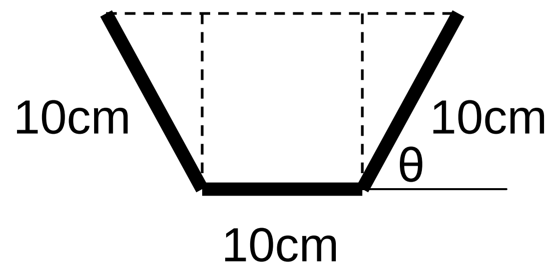

Now let's take the derivative of this area function and find the critical value:

$$A(q) = 100\sin q + 100\cos q \sin q$$

$$A(q) = 100\sin q + 50\sin 2q$$

$$\frac{dA}{dq} = 100\cos q + 100\cos 2q$$

Trig identity:

$$\sin 2q = 2\sin q \cos q$$

$$100\cos q + 100\cos 2q = 0$$

$$100\cos q = -100\cos 2q$$

$$\cos q = -\cos 2q$$

$$\cos q = -(2\cos^2 q - 1)$$

Trig identity:

$$\cos 2q = 2\cos^2 q - 1$$

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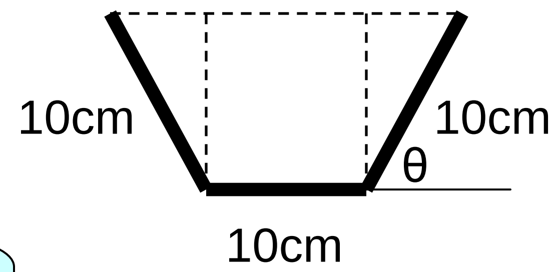

Now let's take the derivative of this area function and find the critical value:

$$\cos q = -(2\cos^2 q - 1)$$

$$\cos q = -2\cos^2 q + 1$$

$$2\cos^2 q + \cos q - 1 = 0$$

Now we can use the quadratic formula to solve for cos *θ*

$$\cos q = \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{2(2)}$$

$$\cos q = \frac{-1 \pm \sqrt{9}}{4}$$

$$\cos q = \left\{-1, \frac{1}{2}\right\}$$

$$\cos^{-1}(-1) = 180^\circ$$

$$\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

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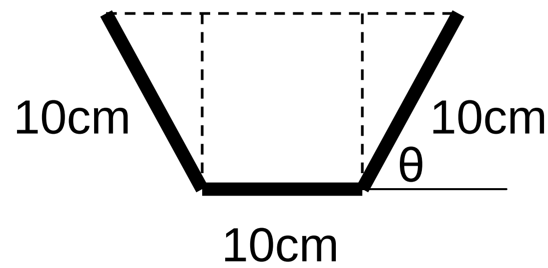


Since we're looking for an angle between 0º and 90º, we can only take θ = 60º. Let's go one step further and find the maximum area:

$$A(q) = 100\sin q + 50\sin 2q$$

$$A(60^\circ) = 100\sin 60^\circ + 50\sin(120^\circ)$$

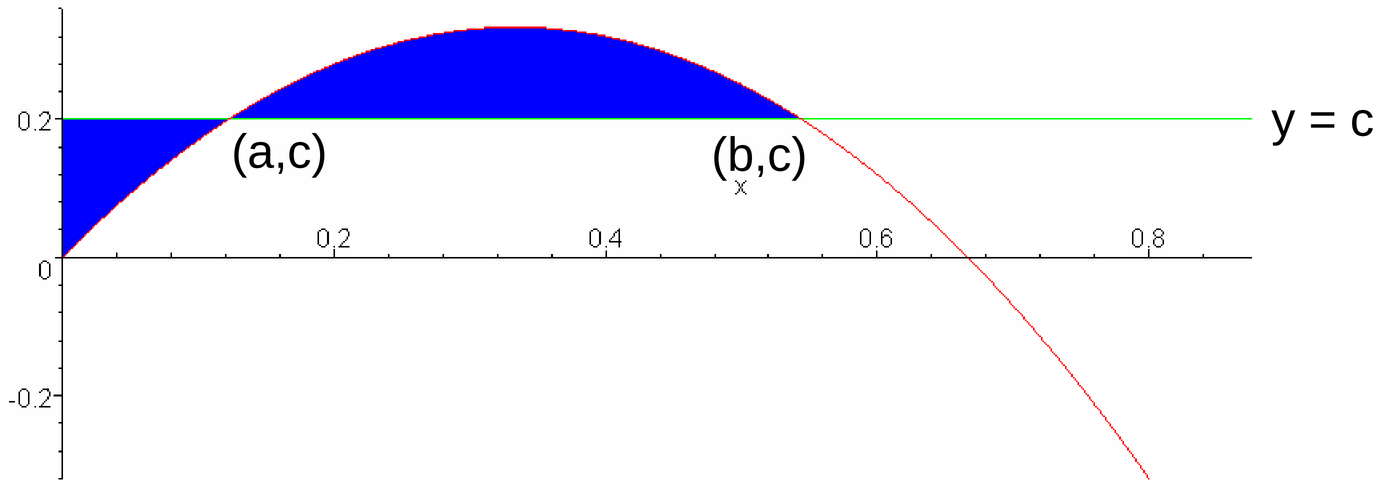
$$A(60^\circ) = 100(3/\sqrt{2}) + 50(3/\sqrt{2})$$

$$A(60^\circ) = 150(3/\sqrt{2})$$

Calculus – Differentiation

The horizontal line y=c intersects the curve $y=2x-3x^2$ in the first quadrant on graph paper. Find c so that the areas of the two shaded regions are equal. The shaded areas included the triangle-like area from (0,0) to (a,c) below y=c & above the parabola, the second shaded area is from (a,c) to (b,c) where the shaded region is the area above y=c but within the parabola. (a,c) and (b,c) are the points of intersection.

Here's an illustration of the areas we to equalize:



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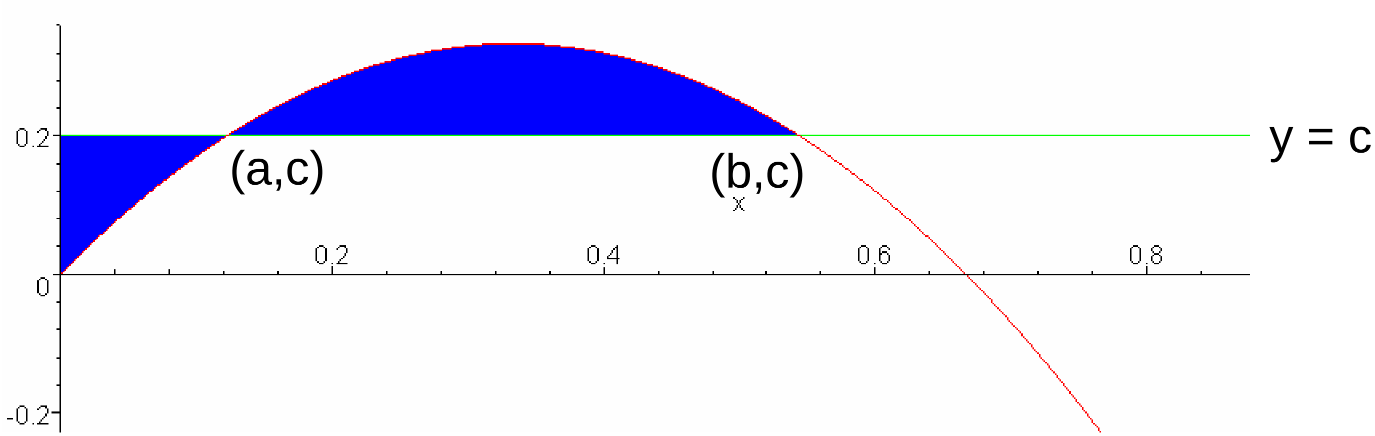
The horizontal line y = c intersects the curve $y=2x-3x^2$ in the first quadrant on graph paper. Find c so that the areas of the two shaded regions are equal. The shaded areas included the triangle-like area from (0,0) to (a,c) below y=c and above the parabola, the second shaded area is from (a,c) to (b,c) where the shaded region is the area above y = c but within the parabola. (a,c) and (b,c) are the points of intersection.

I'm not sure if you've been introduced to this concept, but the only way to find the area beneath curves is via **integration**. (Let me know if you haven't yet learned integration.)



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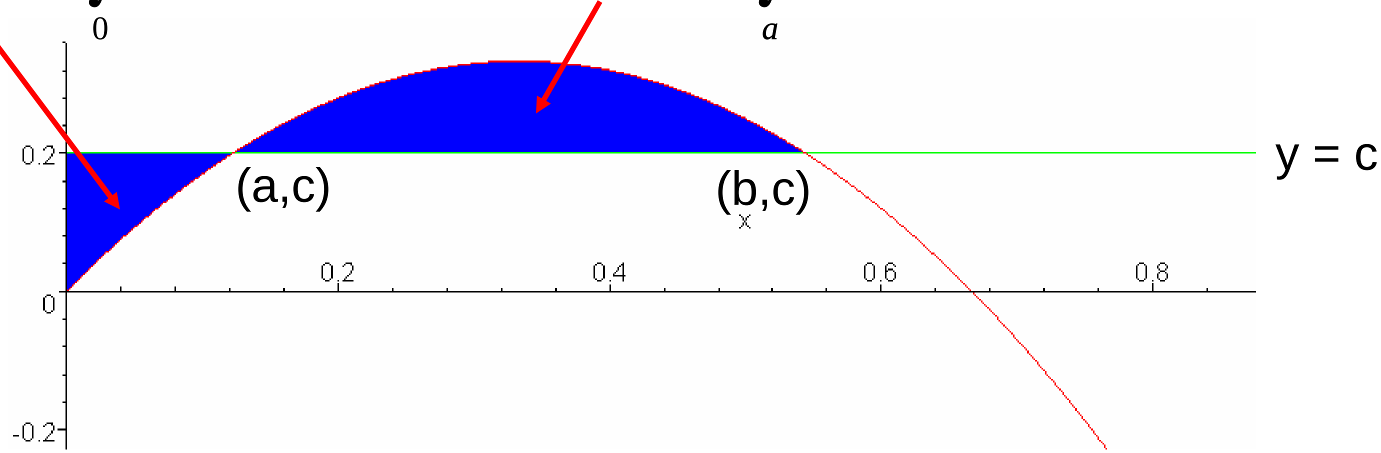
Now let's find a formula for the area of each shaded region using integrals.

$$A_1(x) = \int_0^a [c-(2x-3x^2)]dx \qquad A_2(x) = \int_a^b [(2x-3x^2)-c]dx$$



Calculus – Differentiation

The horizontal line y = c intersects the curve y=2x-3x² in the first quadrant on graph paper. Find c so that the areas of the two shaded regions are equal. The shaded areas included the triangle-like area from (0,0) to (a,c) below y=c and above the parabola, the second shaded area is from (a,c) to (b,c) where the shaded region is the area above y = c but within the parabola. (a,c) and (b,c) are the points of intersection.

Now let's find a formula for the area of each shaded region using integrals.

$$A_1(x) = \int_0^a [c-(2x-3x^2)]dx$$

$$A_1(x) = \left[cx - x^2 + x^3\right]_0^a$$

$$A_1(x) = ca - a^2 + a^3$$

$$A_2(x) = \int_a^b [(2x-3x^2)-c]dx$$

$$A_2(x) = \left[x^2 - x^3 - cx\right]_a^b$$

$$A_2(x) = (b^2 - b^3 - cb) - (a^2 - a^3 - ca)$$

$$A_2(x) = b^2 - b^3 - cb - a^2 + a^3 + ca$$

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So let's compile all the information we know:

$$c = 2x - 3x^2$$

$$c = 2a - 3a^2$$

$$c = 2b - 3b^2$$

These equation come from the intersection points, (a,c) and (b,c)

$$A_1 = A_2$$

This equation comes from equalizing the areas.

$$ca - a^2 + a^3 = b^2 - b^3 - cb - a^2 + a^3 + ca$$

$$0 = b^2 - b^3 - cb$$

Note that b = 0 cannot be a solution, since we presume b > 0

$$0 = b - b^2 - c$$

$$c = b - b^2$$

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Now let's try to solve:

$$c = 2b - 3b^2 = b - b^2$$

$$2b - 3b^2 = b - b^2$$

$$b = 2b^2$$

$$1 = 2b$$

$$b = 1/2$$

$$c = b - b^2$$

$$c = \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$c = \frac{1}{2} - \frac{1}{4}$$

$$c = \frac{1}{4}$$

$$c = 2a - 3a^2$$

$$\frac{1}{4} = 2a - 3a^2$$

$$3a^2 - 2a + 1/4 = 0$$

$$(3a - 1/2)(a - 1/2) = 0$$

$$a = \left\{\frac{1}{6}, \cancel{\frac{1}{2}}\right\}$$

Note a cannot equal ½ since b is already ½