Calculus Problems

§ Find the particular solution of the differential equation that satisfies the boundary condition:

$$5xy' - y = x^3 - 5x$$
$$y\left(\frac{\sqrt{35}}{2}\right) = 0$$

- First convert to standard first-order linear form, which is $y' + p(x)y = q(x)$

  $$5xy' - y = x^3 - 5x$$
  $$y' + \left(-\frac{1}{5x}\right)y = \frac{x^2}{5} - 1$$

  In this case, $p(x) = -\frac{1}{5x}$

- We need to first integrate *p(x)*, and then exponentiate, before multiplying across both sides of the equation.

  $$p(x) = -\frac{1}{5x}$$
  $$\int p(x) = -\frac{1}{5}\ln x = \ln\left(\frac{1}{x^{1/5}}\right)$$
  $$e^{\int p(x)} = e^{\ln\left(\frac{1}{x^{1/5}}\right)} = \frac{1}{x^{1/5}}$$

- Now multiply across both sides of the equation:

  $$5xy' - y = x^3 - 5x$$
  $$\left(\frac{1}{x^{1/5}}\right)y' + \left(\frac{1}{x^{1/5}}\right)\left(-\frac{1}{5x}\right)y = \left(\frac{1}{x^{1/5}}\right)\frac{x^2}{5} - \left(\frac{1}{x^{1/5}}\right)$$
  $$\left(\frac{1}{x^{1/5}}\right)y' + \left(-\frac{1}{5x^{6/5}}\right)y = \frac{x^{9/5}}{5} - \frac{1}{x^{1/5}}$$

- Notice that the left-hand side is the derivative (by product rule) of $\frac{y}{x^{1/5}}$:

  § Product rule: $x^{-1/5}y = x^{-1/5}y' + \left(-\frac{1}{5}x^{-6/5}\right)y$

- Now integrate both side of the equation:

  $$\left(\frac{1}{x^{1/5}}\right)y' + \left(-\frac{1}{5x^{6/5}}\right)y = \frac{x^{9/5}}{5} - \frac{1}{x^{1/5}}$$
  $$\frac{y}{x^{1/5}} = \frac{x^{14/5}}{14} - \frac{5x^{4/5}}{4} + K$$

- Multiply through by $x^{1/5}$ to solve for y:

$$\frac{y}{x^{1/5}} = \frac{x^{14/5}}{14} - \frac{5x^{4/5}}{4} + K$$

$$y = \frac{x^3}{14} - \frac{5x}{4} + Kx^{1/5}$$

- Plug in boundary condition:

$$y = \frac{x^3}{14} - \frac{5x}{4} + Kx^{1/5}$$

$$y = \frac{\left(\sqrt{35}/2\right)^3}{14} - \frac{5\left(\sqrt{35}/2\right)}{4} + K\left(\sqrt{35}/2\right)^{1/5} = 0$$

$$y = \frac{5\sqrt{35}}{16} - \frac{5\sqrt{35}}{8} + K\left(\sqrt{35}/2\right)^{1/5} = 0$$

$$y = \frac{5\sqrt{35}}{16} - \frac{10\sqrt{35}}{16} + K\left(\sqrt{35}/2\right)^{1/5} = 0$$

$$y = -\frac{5\sqrt{35}}{16} + K\left(\sqrt{35}/2\right)^{1/5} = 0$$

$$K = \frac{5(2450)^{1/5}}{16} = 1.489$$

- Assume an object weighing 8 lb is dropped from a height of 8,000 ft, where the air resistance is proportional to the velocity.
  - Write the velocity as a function of time, if its velocity after 4 seconds is 2.00 ft/sec.

    Unless you know some physics, you cannot intuitively derive this equation, but here are the basics:
    - $Force = mass \times acceleration$
    - Acceleration is the derivative of velocity, or $a = \frac{dv(t)}{dt} = v'(t)$
    - Weight $= mass \times gravitational\_acceleration$ [Weight = force due to gravity]
    - Gravitational acceleration (on Earth) $= 32\ ft/s^2$
    - Air resistance opposed gravitational force

    This expression says that the net force on a free-falling object is its weight minus air resistance, which is proportional to velocity:
    $F = mv'(t) = m(32\ ft/s^2) - kv(t)$

    Let's solve for v(t): [we divided through by *m* to get into standard linear form]
    $v'(t) + \frac{k}{m}v(t) = 32\ ft/s^2$
    - $p(t) = \frac{k}{m}$ so $\int p(t) = \frac{k}{m}t$ and $e^{\int p(t)} = e^{\frac{k}{m}t}$
    - Multiplying through: $e^{\frac{k}{m}t}v'(t) + \frac{k}{m}e^{\frac{k}{m}t}v(t) = (32\ ft/s^2)e^{\frac{k}{m}t}$
    - Then integrate: $e^{\frac{k}{m}t}v(t) = (32\ ft/s^2)\frac{m}{k}e^{\frac{k}{m}t} + C$
    - Isolate *v(t)*: $v(t) = (32\ ft/s^2)\frac{m}{k} + Ce^{-\frac{k}{m}t}$

    Now we need to plug in the boundary condition: $v(4) = 2\ ft/s$
    - But first, we know that $v(0) = 0\ ft/s$

$$v(0) = (32\ ft/s^2)\frac{m}{k} + Ce^{-\frac{k}{m}(0)} = 0$$

$$v(0) = (32\ ft/s^2)\frac{m}{k} + C = 0$$

$$C = -(32\ ft/s^2)\frac{m}{k}$$

$$v(0) = (32\ ft/s^2)\frac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

- Now we plug in the point to find *k*, since we know *m* = 8 lb.

$$v(4) = (32\,ft/s^2)\left(\frac{8lb}{k}\right)\left(1 - e^{-\frac{k}{8}(4)}\right) = 2$$

$$v(4) = \left(\frac{256}{k}\right)\left(1 - e^{-\frac{k}{2}}\right) = 2$$

$$1 - e^{-\frac{k}{2}t} = \frac{2k}{256}$$

(It's late but you just need to solve for *k* and plug back into equation)

- What is the limiting value of the velocity function?

  Basically, take the limit as t approaches infinity.

$$v(\infty) = (32\,ft/s^2)\frac{m}{k}\left(1 - e^{-\frac{k}{m}(\infty)}\right)$$

$$v(\infty) = (32\,ft/s^2)\frac{m}{k}(1-0)$$

$$v(\infty) = (32\,ft/s^2)\frac{8lb}{k}$$

  You need *k* to get a numeric answer, but that's really it.